Acta Cryst. (1995). A**51**, 684–690

Chebychev Matrices and Quasiperiodic Tilings

BY DEAN CLARK AND E. R. SURYANARAYAN

Department of Mathematics, University of Rhode Island, Kingston, RI 02881, *USA*

(*Received* 22 *August* 1994; *accepted* 10 *March* 1995)

Abstract

Using 2×2 and 3×3 matrices, Rao & Suryanarayan [*Physica* (*Utrecht*) (1994), B**193**, 139–146] and Clark & Suryanarayan [*Acta Cryst.* (1991), A**47**, 498–502] have obtained quasiperiodic tilings of the plane with n-fold rotational symmetry, $n = 2, 3, 4, 5, 6, 8$ with two unit prototiles. In this paper, a generalized method for generating quasiperiodic lattices for n-fold non-crystallographic axes is given by employing Chebychev and associated Chebychev matrices of order n, and some of their properties are derived. The method is based on the self-similarity principle. The properties of the matrices are applied to create self-similar tiles by solving an eigenvalue problem that shows how many of each type of tile to use and sheds light on how to configure the boundaries of the next generation's tiles. The tilings generated contain the above-mentioned tilings as special cases. Thus, this approach introduces the basic techniques from linear algebra to the study of tilings.

1. Introduction

Chebychev polynomials u_n of the second kind are best known from classical analysis, part of a huge body of knowledge of families of orthogonal polynomials arising most often in the constructive theory of functions. The classic reference is Szegö (1959). It is said that the Chebychev polynomial is like a fine jewel that reveals its different characteristics under illumination from varying positions (Rivlin, 1974). There is yet another area where it shows its brilliance: quasiperiodic tilings.

Initiating the inflation method, Penrose (1974) showed how to tile the Euclidean plane with 5-fold symmetry aperiodically, using the two rhombi whose vertex angles are $\pi/5$ and $2\pi/5$. A decade later, when Shechtman, Blech, Gratias & Cahn (1984) discovered that rapidly solidified Al–Mn alloys exhibited the forbidden 5-fold symmetry in electron diffraction patterns, Penrose tilings became possible models (Levine & Steinhardt, 1984; Gratias & Michel, 1986) for structural ordering in quasicrystals. Now several methods are available to tile a plane aperiodically with 5-fold symmetry. One of them is the generalized projection method initiated by de Bruijn (1981), who showed that the Penrose patterns could be derived by projecting a 5-dimensional cubic lattice, and suggested that the process could be generalized. Using the projection method of de Bruijn (1981), Whittaker & Whittaker (1988) obtained nonperiodic tilings with n-fold symmetry with $n = 5, 7, 8, 9, 10$ and 12. However, when their method was applied to the cases of 3-, 4- and 6-fold rotational symmetry, it produced periodic tilings. Clark & Suryanarayan (1991) constructed quasiperiodic tilings with 3-, 4- and 6-fold rotational symmetry using the inflation method, proving that the 3-, 4- and 6-fold symmetry is compatible with nonperiodicity. The inflation method was also used by Watanabe, Ito & Soma (1987) to build a tiling with 8-fold symmetry; the same tiling was obtained by Whittaker & Whittaker (1988) using the projection method. Balagurusamy, Ramesh & Gopal (1992) have recently constructed 5-fold and 10-fold nonperiodic tilings using a particular arrangement of rhombi of the previous generation.

In this paper, we develop a method that is related to Penrose's inflation method. However, we use a matrix eigenvalue approach to determine each tiling, thus providing a mathematical basis for the inflation (deflation) process. We do this by considering the properties of the Chebychev polynomial of the second kind, $u_n(x)$, and the matrix associated with it, $\mathbf{U}_n(x)$. Related to $\mathbf{U}_{2k}(x)$ and $\mathbf{U}_{2k+1}(x)$ are two matrices of order k, $\mathbf{A}_k(x)$ and $\mathbf{B}_k(x)$, whose eigenvalues are related to the combinatorics of quasiperiodic tilings of the Euclidean plane. In fact, the eigenvalues of $\mathbf{A}_k$ and $\mathbf{B}_k$ define k kinds of rhombi that turn out to be basic building blocks, *prototiles* (Grünbaum & Shephard, 1987). Thus, each matrix provides a natural setting for generating a class of nonperiodic tessellations of the plane, which contain, among others, the tilings generated by Balagurusamy *et al.* (1992), Clark & Suryanarayan (1991), Penrose (1974) and Watanabe *et al.* (1987). The two matrices not only define the k parent cells but also furnish a formula for self-similar subdivision (or build-up) of the parent cells into smaller (or larger) rhombi, thus giving a tessellation. In a self-similar subdividing operation, the underlying question is how many unit rhombi (rhombi with edge length one) of each type are required to pack the two first-generation cells? The answer lies in solving the underlying inflation (or deflation) problem (Clark & Suryanarayan, 1991).

When tiling a plane, one has to make sure that the edges of the tiles match along the boundaries with their immediate neighbors. Therefore, in generating a larger replica of a tile from one generation to the next, one has to define matching conditions (forcing rules) for the edges which have to be obeyed all through the construction of the tiling. Many a time one has to go back and forth using trial-and-error methods to define a matching condition that is satisfied by the tiling. For example, in order to use first-generation tiles as building blocks for self-similar second-generation tiles, we must be sure that the half-rhombi along the edges agree, thus restoring the whole rhombi. It is not obvious that the matching conditions will be fulfilled simply by replacing a prototile with the corresponding first-generation tiles. However, in one instance it is fairly obvious because the edge sequence has the useful property of palindromy (the property that the edge length of each rhombus of one generation is made up of a sequence of edges and diagonals of rhombi of its previous generation such that the sequence reads the same backwards as forwards, as in Figs. 2 and 3). Thus, all the rhombi of the same generation match along their common boundaries, forcing quasiperiodicity. Even though there are infinitely many ways to arrange the tiles with identical edge sequencies, Balagurusamy *et al.* (1992) have generated quasiperiodic tilings in which the decision must be made for the first generation only, and subsequent generations follow the decisions made in the first generation.

2. Generating method

Even though the problem of tiling is ancient, some striking developments have occurred since the 1970s (Grünbaum & Shepherd, 1987). Recent work in quasicrystals has emphasized 5-fold rotational symmetry for good empirical and theoretical reasons. In this paper, we use unit rhombi defined by the k components of the eigenvector associated with the largest eigenvalue of $\mathbf{A}_k$ (or $\mathbf{B}_k$) as prototiles. The identities

$$2n\left[\sum_{j=1}^{n}\sin(j\pi/2n+1/2)\right]=n\cot(\pi/4n),$$

$$(2n+1)\left[\sum_{j=1}^{n}\sin(j\pi/2n+1)\right]$$

$$=[(2n+1)/4]\cot\pi/[2(2n+1)]$$

are the immediate consequencies of the property that a regular convex polygon of $4n$ or $4n+2$ can be decomposed into n kinds of prototiles with vertex angles of $j\pi/2n$ or $j\pi/(2n+1)$, respectively (see Fig. 1). These prototiles are used as building blocks by placing them edge-to-edge, without overlaps or gaps, in infinite assemblies called tilings, so that the Euclidean plane is covered. A rhombus may be cut along a diagonal as long as the cut diagonal edge appears on the outer boundary of the tile; this is in anticipation of matching the half-rhombus with a half-rhombus belonging to another tile, thus producing a full rhombus. For example, in Figs. 2(*a*)–(*c*), only the $2\pi/6$ rhombus is cut along its larger diagonal; similarly, in Figs. 3(*a*)–(*c*), only the $2\pi/7$ rhombus is cut along its larger diagonal. Thus, the rule for cutting a rhombus along a diagonal is dictated by the fact that the first-generation tiles (as well as the succeeding generations) will also be rhombi, that is, larger replicas built from the prototiles.

3. Chebychev matrices

Chebychev polynomials of the second kind are defined by (Rivlin, 1974)

$$x=\cos\theta,\quad u_n(x)=\sin(n+1)\theta/\sin\theta. \tag{1}$$

From this, $u_0(x)=1$, $u_1(x)=2x$ and by applying the trignometric identities, we get the classical recurrence

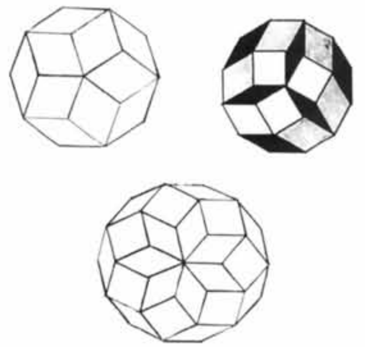


Fig. 1. Regular polygons of 10, 12 and 14 sides and their prototiles.

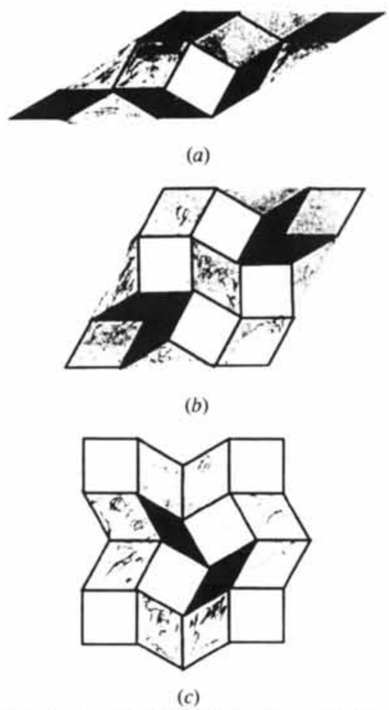


Fig. 2. The first-generation tiles defined by $\mathbf{B}_3(1)$.

relation

$$u_n = 2xu_{n-1} - u_{n-2}, \quad u_0 = 1, \quad u_1 = 2x, \ldots . \tag{2}$$

It also follows that these polynomials can also be expressed by the relation $u_n(x) = \det \mathbf{U}_n(x)$, where $\mathbf{U}_n(x)$ denotes the $n \times n$ tridiagonal symmetric matrix

$$\mathbf{U}_n(x) = \begin{bmatrix} 2x & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2x & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2x \end{bmatrix}. \tag{3}$$

There is another, much less known, recurrence relation that also generates the Chebychev polynomials:

Theorem 1

$$u_{m+n}(x) = \det \begin{bmatrix} u_m(x) & u_{m-1}(x) \\ u_{n-1}(x) & u_n(x) \end{bmatrix}, \quad m, n = 1, 2, \ldots . \tag{4}$$

Proof

From (1), we get

$$\sin^2\theta[u_m(x)u_n(x) - u_{m-1}(x)u_{n-1}(x)] = \sin(m+1)\theta \sin(n+1)\theta - \sin m\theta \sin n\theta, \tag{5}$$

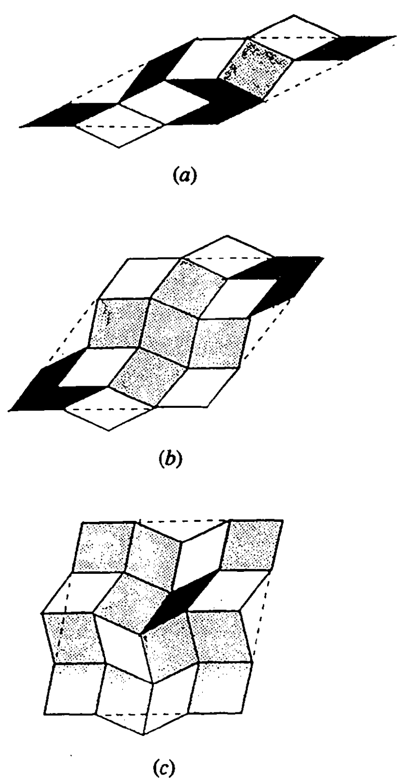


Fig. 3. The first-generation tiles defined by $A_3(1)$.

which, by further application of trignometric identities, reduces to

$$\sin\theta[u_m(x)u_n(x) - u_{m-1}(x)u_{n-1}(x)] = \sin(m+n+1)\theta = \sin\theta[u_{m+n}(x)]. \quad \blacksquare$$

The following 'odd–even' breakdown will be fundamental in the sequel.

Corollary 1

$$u_{2n} = (u_n + u_{n+1})(u_{n+1} - u_{n-1}), \quad n = 1, 2, \ldots . \tag{6}$$

Proof

Set $m = n$ in (4). ■

Corollary 2

$$u_{2n+1} = u_n(u_{n+1} - u_{n-1}), \quad n = 1, 2, \ldots . \tag{7}$$

Proof

Set $m = n + 1$ in (4). ■

Theorem 2

The eigenvalues of $\mathbf{U}_n(x)$ are $\lambda_j = 2x + 2\cos[j\pi/(n+1)]$, with associated eigenvectors $\mathbf{v}_j$, where $(\mathbf{v}_j)_i = \sin[ij\pi/(n+1)]$, $j = 1, 2, \ldots, n$.

Proof

Consider the ith element $(\mathbf{U}_n\mathbf{v}_j)_i$ in the product $\mathbf{U}_n\mathbf{v}_j$, $i = 2, \ldots, n-1$,

$$(\mathbf{U}_n\mathbf{v}_j)_i = \sin\frac{(i-1)\pi}{n+1} + 2x\sin\frac{ij\pi}{n+1} + \sin\frac{(i+1)j\pi}{n+1}. \tag{8}$$

(Note that the above relation is also valid for $i = 1, n$.) Using the identity $\sin(a-b) + \sin(a+b) = 2\sin a\cos b$ with $a = [ij\pi/(n+1)]$, $b = [j\pi/(n+1)]$ gives

$$(\mathbf{U}_n\mathbf{v}_j)_i = 2\sin\frac{ij\pi}{(n+1)}\cos\frac{j\pi}{(n+1)} + 2x\sin\frac{ij\pi}{(n+1)} = (\lambda_j\mathbf{v}_j)_i, \quad j = 1, \ldots, n. \quad \blacksquare \tag{9}$$

We now interpret geometrically the largest eigenvalue of $\mathbf{U}_n(x)$, $\lambda_1 = 2x + 2\cos[\pi/(n+1)]$. When $2x$ is a non-negative integer, λ_1 is an integer length $2x$ plus the length of the long diagonal, $2\cos[\pi/(n+1)]$, of a unit-edged rhombus with vertex angle $2\pi/(n+1)$. The elements of the associated eigenvector $\mathbf{v}_1$ are *areas* of unit rhombi with vertex angle $j\pi/(n+1)$, $j = 1, \ldots, n$. Thus, the relation $\mathbf{U}_n(x)\mathbf{v}_1$ suggests that unit rhombi can be combined as building blocks, the precise numbers determined by the elements of $\mathbf{U}_n$, into larger rhombi with edge length λ_1. The edges of the larger rhombi are built from the edges of $2x$ unit rhombi plus one half-rhombus cut along its diagonal. Details of the construction follow. First, a minor inconvenience: the non-distinctness of the elements of $\mathbf{v}_1$. From a geometric point of view, $\sin[\pi/(n+1)] = \sin[n\pi/(n+1)]$, $\sin[2\pi/(n+1)] = \sin[(n-1)\pi/(n+1)]$ *etc.* refer to the same rhombi from two different vertex angles. We could get around this by adding the columns of $\mathbf{U}_n$ (last to first, next-to-last to second *etc.*), thus compressing $\mathbf{U}_n$ into a matrix with eigenvectors consisting of distinct members. We prefer a less heavy handed approach, letting the factorizations of the corollaries guide the way.

The corollaries suggest that u_n is always the product of two determinants which are themselves 'Chebychev polynomials' in the sense that $u_k + u_{k-1}$, $u_k - u_{k-1}$ and $u_{k+1} - u_{k-1}$ satisfy the recurrence (2). Indeed,

$$u_k \pm u_{k-1} = 2xu_{k-1} - u_{k-2} \pm u_{k-1} = (2x \pm 1)u_{k-1} - u_{k-2} = \det\begin{bmatrix} 2x & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2x & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2x \pm 1 \end{bmatrix} \quad (10)$$

and

$$u_{k+1} - u_{k-1} = 2xu_k - 2u_{k-1} = \det\begin{bmatrix} 2x & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2x & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & 1 \\ 0 & 0 & 0 & \cdots & 2 & 2x \end{bmatrix} \quad (11)$$

differ from (1) at just one position. Therefore, we focus on the two matrices (10) and (11) above. For $n = 2k + 1$, let $\mathbf{A}_k$ denote the $k \times k$ matrix

$$\mathbf{A}_k = \begin{bmatrix} 2x & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2x & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & 1 \\ 0 & 0 & 0 & \cdots & 2 & 2x+1 \end{bmatrix}, \quad n = 1, 3, 5, \ldots. \quad (12)$$

For $n = 2k$, let $\mathbf{B}_k$ denote the $k \times k$ matrix

$$\mathbf{B}_k = \begin{bmatrix} 2x & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2x & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2x & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 2x & 1 \\ 0 & \cdots & 0 & 0 & 2 & 2x \end{bmatrix}, \quad n = 2, 4, 6, \ldots. \quad (13)$$

All statements with $\mathbf{A}_k$ (or $\mathbf{B}_k$) assume that $n = 2k + 1$ (or $n = 2k$).

Theorem 3

If $\lambda = 2x + 2\cos(\pi/n)$ and $\mathbf{v}_i = \sin(i\pi/n)$, then $\mathbf{A}_k\mathbf{v} = \lambda\mathbf{v}$ and $\mathbf{B}_k\mathbf{v} = \lambda\mathbf{v}$.

Proof

If we consider the ith element $(\mathbf{A}_k\mathbf{v})_i$ in the product $\mathbf{A}_k\mathbf{v}$, $i = 1, \ldots, k-1$, we essentially get a repetition of (8) and (9) with $i = 1$. When $i = k$, noting that $\sin(k\pi/n) = \sin[(k+1)\pi/n]$, we have

$$\begin{aligned}(\mathbf{A}_k\mathbf{v})_k &= \sin[(k-1)\pi/n] + (2x+1)\sin(k\pi/n) \\ &= \sin[(k-1)\pi/n] + 2x\sin(k\pi/n) + \sin(k\pi/n) \\ &= 2\sin(k\pi/n)\cos(\pi/n) + 2x\sin(k\pi/n) \\ &= (\lambda\mathbf{v})_k.\end{aligned}$$

Similarly, we have only to show that $(\mathbf{B}_k\mathbf{v})_k = (\lambda\mathbf{v})_k$. But $(\mathbf{B}_k\mathbf{v})_k = 2\sin[(k-1)\pi/n] + 2x\sin(k\pi/n)$ leads to a repetition of the above result. ■

Thus, $\mathbf{A}_k$ and $\mathbf{B}_k$ not only define the k unit prototiles but also determine the self-similar replicas of the prototiles that tile the plane.

Using the above results, let us construct some quasiperiodic tilings. Tilings produced by $\mathbf{B}_2$ and $\mathbf{A}_2$ have already been studied in considerable detail (Clark & Suryanarayan, 1991; Balagurusamy *et al.*, 1992; Rao & Suryanarayan, 1994). To build tilings generated by $\mathbf{A}_3(x)$ and $\mathbf{B}_3(x)$, let us first consider the sequence $\mathbf{B}_3(x)$ for $x = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

$$\mathbf{B}_3(x) = \begin{bmatrix} 2x & 1 & 0 \\ 1 & 2x & 1 \\ 0 & 2 & 2x \end{bmatrix}. \quad (14)$$

Its eigenrelations are

$$\begin{aligned}&\mathbf{B}_3(x)[\sin(\pi/6), \sin(2\pi/6), \sin(3\pi/6)]^T \\ &\quad = [2x + 2\cos(\pi/6)][\sin(\pi/6), \sin(2\pi/6), \sin(3\pi/6)]^T, \\ &\mathbf{B}_3^2(x)[\sin(\pi/6), \sin(2\pi/6), \sin(3\pi/6)]^T \\ &\quad = [2x + 2\cos(\pi/6)]^2[\sin(\pi/6), \sin(2\pi/6), \sin(3\pi/6)]^T.\end{aligned} \quad (15)$$

The last equation is a combinatorial identity, *area identity*, which reveals how many unit rhombi of each type to use in building larger replicas with edge length $\lambda(x) = 2x + 2\cos(\pi/6)$. In fact, the exact numbers of each rhombus can be read from the rows of $\mathbf{B}_3^2(x)$:

$$\mathbf{B}_3^2(x) = \begin{bmatrix} 4x^2+1 & 4x & 1 \\ 4x & 4x^2+3 & 4x \\ 2 & 8x & 4x^2+2 \end{bmatrix}. \quad (16)$$

Let r_1, r_2 and r_3 denote the $(\pi/6)$-, $(2\pi/6)$- and $(3\pi/6)$-unit rhombi, respectively. We can then formally write (16) as

$$(4x^2+1)r_1 + 4xr_2 + r_3 = \lambda(x)r_1, \quad (17)$$

$$4xr_1 + (4x^2+3)r_2 + 4xr_3 = \lambda(x)r_2, \quad (18)$$

$$2r_1 + 8xr_2 + (4x^2+2)r_3 = \lambda(x)r_3, \quad (19)$$

where $\lambda(x)$ is the expansion factor. In particular, if $x = 1$, then the above set of identities becomes

$$5r_1 + 4r_2 + r_3 = \lambda(1)r_1, 4r_1 + 7r_2 + 4r_3 = \lambda(1)r_2, 2r_1 + 8r_2 + 6r_3 = \lambda(1)r_3.$$

The above relation tells us how the first-generation tiles are built from the prototiles; The first-generation r_1 tile is packed with five r_1-prototiles, four r_2-prototiles and one r_3-prototiles, and so on. Fig. 2 shows the construction of the first-generation tiles from the prototiles defined by $\mathbf{B}_3(1)$. At first glance, the projections and indentations in Figs. 2(*c*) and 3(*c*) look like possible mismatches, but in fact each is simply a generalized rhombus, which is a new aspect of the solution. The reader will see how the projections and indentations work in Figs. 4 and 5.

Similarly, we can build the tiling defined by the matrix $\mathbf{A}_3(x)$. The sequence $\mathbf{A}_3(x)$, $x = 0, 1/2, 1, 3/2, \ldots$ yields

$$\mathbf{A}_3(x) = \begin{bmatrix} 2x & 1 & 0 \\ 1 & 2x & 1 \\ 0 & 1 & 2x+1 \end{bmatrix}, \ldots.$$

Its eigenrelations are

$$\mathbf{A}_3(x)[\sin(\pi/7), \sin(2\pi/7), \sin(3\pi/7)]^T = [2x + 2\cos(\pi/7)][\sin(\pi/7), \sin(2\pi/7), \sin(3\pi/7)]^T,$$

$$\mathbf{A}_3^2(x)[\sin(\pi/7), \sin(2\pi/7), \sin(3\pi/7)]^T = [2x + 2\cos(\pi/7)]^2[\sin(\pi/7), \sin(2\pi/7), \sin(3\pi/7)]^T. \quad (20)$$

The last equation is a combinatorial identity which reveals how many unit rhombi of each type to use in building larger replicas with edge length $\lambda(x) = 2x + 2\cos(\pi/7)$. In fact, the exact numbers of each rhombus can be read from the rows of $\mathbf{A}_3^2(x)$:

$$\mathbf{A}_3^2(x) = \begin{bmatrix} 4x^2+1 & 4x & 1 \\ 4x & 4x^2+2 & 4x+1 \\ 1 & 4x+1 & 4x^2+4x+2 \end{bmatrix}. \quad (21)$$

If s_1, s_2 and s_3 denote the $(\pi/7)$-, $(2\pi/7)$- and $(3\pi/7)$-unit rhombi, respectively, we can then formally write (21) in the form

$$(4x^2+1)s_1 + 4xs_2 + s_3 = \lambda(x)s_1, \quad (22)$$

$$4xs_1 + (4x^2+2)s_2 + (4x+1)s_3 = \lambda(x)s_2, \quad (23)$$

$$s_1 + (4x+1)s_2 + (4x^2+4x+2)s_3 = \lambda(x)s_3, \quad (24)$$

where $\lambda(x) = 2x + 2\cos(\pi/7)$ is the expansion factor. In particular, if $x = 1$, then the above set of identities becomes

$$5s_1 + 4s_2 + s_3 = \lambda(1)s_1, 4s_1 + 6s_2 + 5s_3 = \lambda(1)s_2, 1s_1 + 5s_2 + 10s_3 = \lambda(1)s_3.$$

Fig. 3 shows the construction of the first-generation tiles from the prototiles defined by $\mathbf{A}_3(1)$. As we mentioned earlier, the eigenrelations are *area identities* which give the exact number of rhombi of each type to use in building larger replicas; and the rows of $\mathbf{A}_3^2$ reveal the exact numbers of rhombi of each type to use in building larger (or smaller) copies. However, in one case it is clear that edges agree (by induction because the edge sequence has the property of palindromy). To tile the plane we may use the basic models in Figs. 4(*a*)–(*c*), with 3-, 6- and 12-fold rotational symmetries, respectively, and Figs. 5(*a*) and (*b*) with 7- and 14-fold rotational symmetry and pass to the limit as *n* approaches infinity. Since our method is an instance of a composition process (Grünbaum & Shepherd, 1987), we refer the reader to various approaches to prove that such tilings must be nonperiodic. It is intuitively clear, for now, that the limiting tilings of figures have a unique point of rotational symmetry; thus, we are assured of nonperiodicity because of the lack of translational symmetry.

Figs. 6(*a*) and (*b*) show that the palindromic property of the edge sequences in $\mathbf{A}_3(1)$ and $\mathbf{B}_3(1)$ also enables us to tile the surfaces of right circular cones and right circular cylinders.

Since $\mathbf{A}$ can be diagonalized, we have $\mathbf{A} = \mathbf{SLS}^{-1}$, where

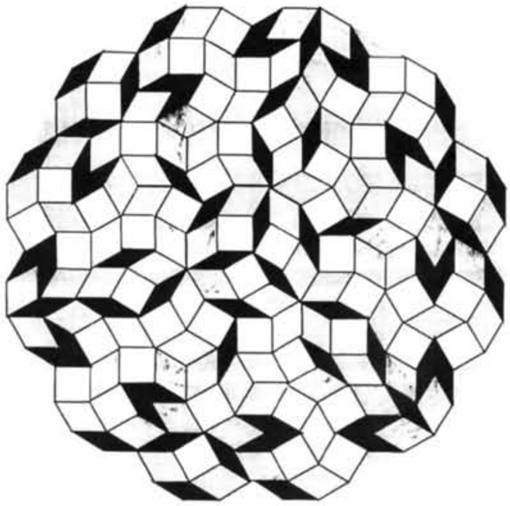
$$\mathbf{S} = \begin{bmatrix} \sin(\pi/7) & \sin(2\pi/7) & \sin(3\pi/7) \\ \sin(2\pi/7) & \sin(4\pi/7) & \sin(6\pi/7) \\ \sin(3\pi/7) & \sin(6\pi/7) & \sin(9\pi/7) \end{bmatrix}$$

and

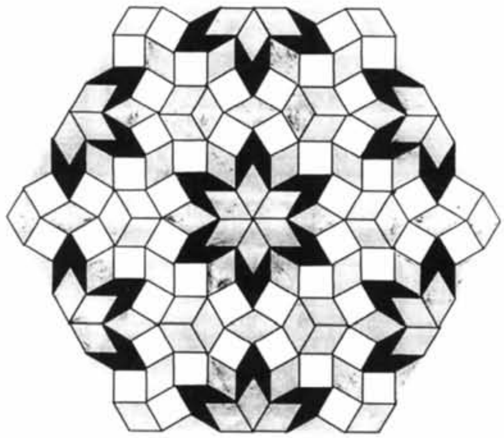
$$\mathbf{L} = \begin{bmatrix} 2x+2\cos(\pi/7) & 0 & 0 \\ 0 & 2x+2\cos(2\pi/7) & 0 \\ 0 & 0 & 2x+2\cos(3\pi/7) \end{bmatrix}.$$

The above result enables us to compute $\mathbf{A}^{2m} = \mathbf{SL}^{2m}\mathbf{S}^{-1}$ and determine the numbers of zero-generation tiles (prototiles) of each kind required to pack the *m*th-generation tiles. We have similar expression for $\mathbf{B}^{2m}$ to compute the number of prototiles of each kind required to pack the *m*th-generation tiles.

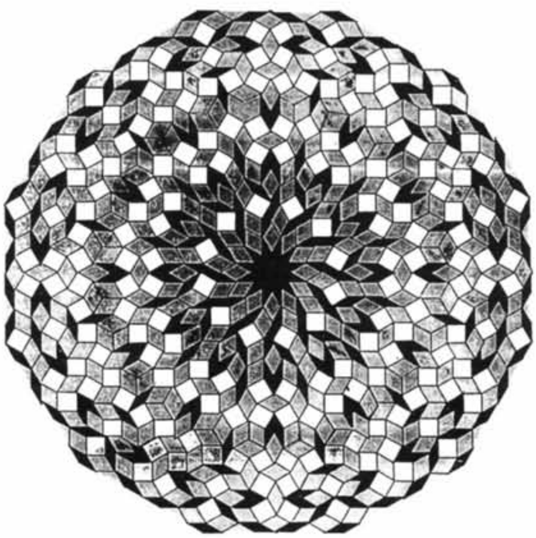
In the general case, each matrix $\mathbf{A}_k(x)$ and $\mathbf{B}_k(x)$ defines k prototiles. In the particular case for $x = 1$, $\mathbf{A}_k(1)$ and $\mathbf{B}_k(1)$ define prototiles with palindromic edge sequences. The construction of the first-generation tiles for these is similar to the first-generation tiles we have built for the tiles defined by $\mathbf{A}_3(1)$ and $\mathbf{B}_3(1)$.



(*a*)

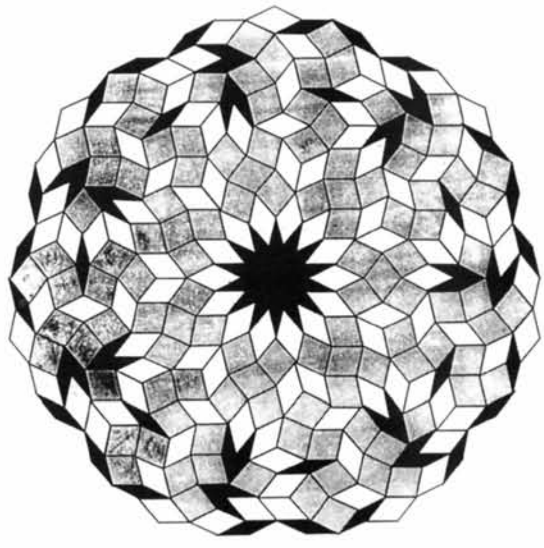


(*b*)

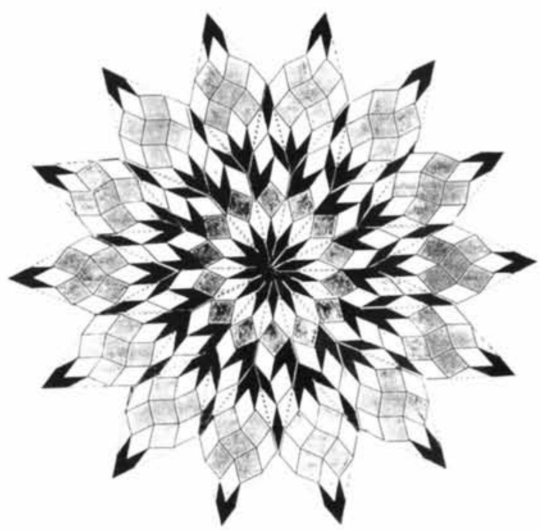


(*c*)

Fig. 4. (*a*) 3-fold rotational symmetry, second generation. (*b*) 6-fold rotational symmetry, second generation. (*c*) 12-fold rotational symmetry, second generation.



(*a*)

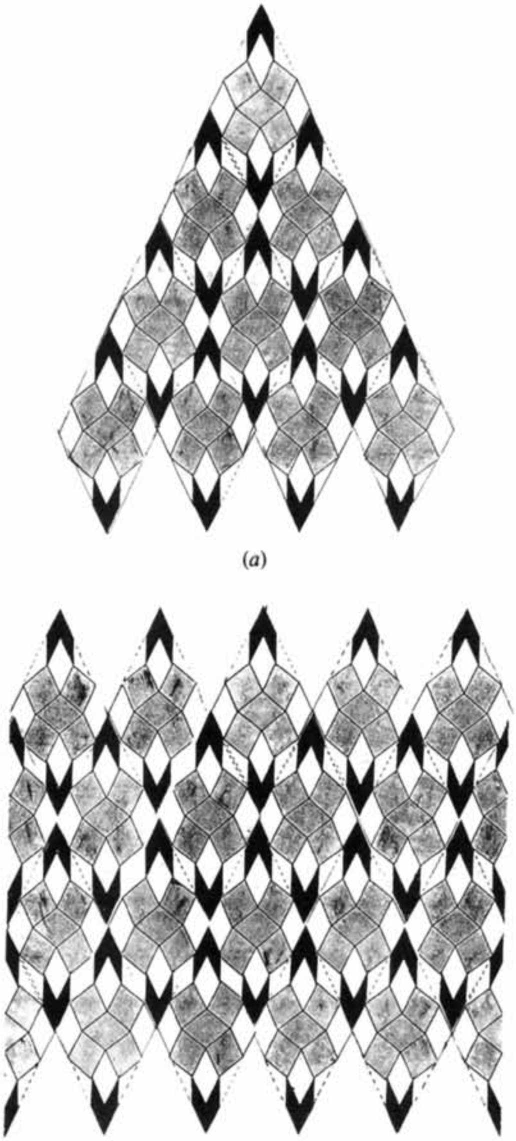


(*b*)

Fig. 5. (*a*) 7-fold rotational symmetry, second generation. (*b*) 14-fold rotational symmetry, second generation.

4. Concluding remarks

The tilings defined by $\mathbf{A}_k(1)$ and $\mathbf{B}_k(1)$ described above are quasiperiodic following the definition of Janssen (1988) and the arguments put forth by Balagurusamy *et al.* (1992). Now a word about the 'class identifier' x that appears in the matrices $\mathbf{A}_k(x)$ and $\mathbf{B}_k(x)$. It determines (*a*) the inflation factor λ and (*b*) the precise numbers of rhombi required to build the next generation rhombi. Watanabe *et al.* (1987) calculated the diffraction pattern in the case of the $\pi/4$ tile defined by $\mathbf{B}_2(1)$, using a FFT algorithm about the center of the $\pi/4$ rhombus by placing unit scatterers at the 1423 vertices. The diffraction pattern showed sharp Bragg-like peaks with 8-fold symmetry as we should expect, because the tiling with 8-fold symmetry is built with eight $\pi/4$ prototiles at the center. Similarly, the diffraction patterns for the *m*th-generation tiles, $\pi/6$, $2\pi/6$ and $3\pi/6$ defined by $\mathbf{B}_3(1)$ should exhibit Bragg-like peaks with 12-fold symmetry, 6-fold symmetry and a 4-fold symmetry, respectively. Again, for the *m*th-generation tiles defined by $\mathbf{A}_3(1)$, the $\pi/7$ rhombus should exhibit 14-fold symmetry and the $2\pi/7$ rhombus should exhibit a 7-fold symmetry. These results can also be extended for the tiles defined by $\mathbf{A}_k(x)$ and $\mathbf{B}_k(x)$.



(*a*)

(*b*)

Fig. 6. Tiling on (*a*) a right circular cone and (*b*) a right circular cylinder.

We could generalize our results to tiling a *p*-dimensional space, by extending the eigenrelations:

$$\mathbf{A}_k^p(x)(s_1, s_2, \ldots, s_k) = \{2x + 2\cos[\pi/(2n+1)]\}^p \times (s_1, s_2, \ldots, s_k), \quad (25)$$

$$\mathbf{B}_k^p(x)(r_1, r_2, \ldots, r_k) = [2x + 2\cos(\pi/2n)]^k \times (r_1, r_2, \ldots, r_k), \quad (26)$$

where $s_j = \sin[j\pi/(2n+1)]$ and $r_j = \sin(j\pi/2n)$.

However, the difficulty in working in three or higher dimensions lies in determining the shapes of the prototiles and how they should be cut and fit to produce larger replicas of themselves. This is the subject of another paper.

References

Balagurusamy, V. S. K., Ramesh, K. & Gopal, E. S. R. (1992). *Physica (Utrecht)*, B**179**, 249–256.

Bruijn, N. G. de (1981). *Proc. K. Ned. Akad. Wet. Ser. A*, **43**, 39–66.

Clark, D. S. & Suryanarayan, E. R. (1991). *Acta Cryst.* A**47**, 498–502.

Gratias, D. & Michel, L. (1986). *Proc. Int. Conf. on Aperiodic Crystals, Les Houches, France. J. Phys. (Paris) Colloq.* **47**, C3.

Grünbaum, B. & Shepherd, G. C. (1987). *Tilings and Patterns.* New York: W. H. Freeman.

Janssen, T. (1988). *Phys. Rep.* **168**, 55–113.

Levine, D. & Steinhardt, P. J. (1984). *Phys. Rev. Lett.* **53**, 2477–2480.

Penrose, J. (1974). *J. Inst. Math. Appl.* **10**, 266–271.

Rao, S. N. & Suryanarayan, E. R. (1994). *Physica (Utrecht)*, B**193**, 139–146.

Rivlin, T. J. (1974). *The Chebychev Polynomials.* New York: John Wiley.

Shechtman, D., Blech, I., Gratias, D. & Cahn, J. W. (1984). *Phys. Rev. Lett.* **53**, 1951–1953.

Szegö, G. (1959). *Orthogonal Polynomials. Am. Math. Soc. Colloq.* Vol. 23.

Watanabe, Y., Ito, M. & Soma, T. (1987). *Acta Cryst.* A**43**, 133–134.

Whittaker, E. J. W. & Whittaker, R. M. (1988). *Acta Cryst.* A**44**, 105–112.